The Handshake Puzzle

Introduction

The handshake puzzle is a classic mathematical problem that involves finding the total number of handshakes between finite numbers of people. This puzzle is rooted in an important area of mathematics known as combinatorics, which concerns the study of combinations, permutations, and enumeration of elements within a finite set. It is therefore often considered 'the art of counting'. Whilst aspects of combinatorics are believed to date back to ancient China and India, it was only in the 17th century when the subject significantly began to develop. In fact, much of its development is accredited to well-known mathematicians of the time - **Pascal**, **Fermat** and **Euler** - each of whom discovered important combinatorial results, which contributed to the development of probability theory. More recent advancement in combinatorics is attributed to the growth of computer science and the need for algorithmic methods to solve real-world problems. These have led to applications in a wide range of areas including coding theory, experimental design, and DNA sequencing.

Many fields in mathematics have strong foundations in combinatorics such as graph theory, probability, and statistics. In particular, the "N choose 2" formula, which serves as the basis of the handshake puzzle, is a fundamental concept in probability that can be used to compute a wide range of mathematical problems, as will be demonstrated in this workshop.

Aim of Workshop

The aim of this workshop is to introduce students to the basic concepts in combinatorics, whilst also developing their problem-solving skills through induction and through recognising patterns. Students will be provided with the opportunity to simulate the handshake puzzle in an effort to find a general formula for the problem and also contribute to the development of their team-work and communication skills.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain, in their own words, what is meant by combinatorics
- Calculate the total number of handshakes between any given number of people
- Justify why the general formula for the total number of handshakes is

$$\frac{n(n-1)}{2}$$

- Apply the "N choose 2" formula to various mathematical problems

The Handshake Puzzle: Workshop Outline

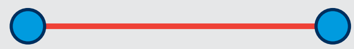
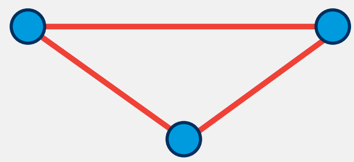
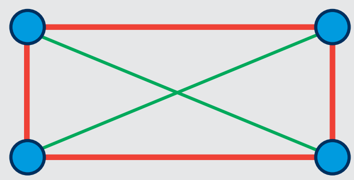
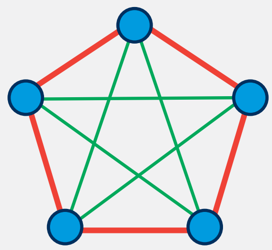
SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
2–5 mins (00:05)	**Introduction to the Handshake Puzzle**	– Introduce the handshake puzzle and outline the rules: 1. Each person must shake hands with every other person in the room 2. Each pair shake hands only once 3. A person cannot shake hands with themselves (Note: Students will be unable to do the activity without these rules)
15 mins (00:20)	**Activity 1** **The Handshake Puzzle**	– **Activity Sheet 1**: Students are asked to determine the number of handshakes between given numbers of people (see **Appendix – Note 1**) – Encourage students to draw diagrams and/or simulate the puzzle in small groups – Facilitate a whole class discussion on students' solutions. – You may wish to use the online simulator as an additional visualisation for students (see link in **Additional Resources**)
10–15 mins (00:35)	**Activity 2** **The Handshake Puzzle**	– **Activity Sheet 2:** Students now attempt to find a general formula for the total number of handshakes (see **Appendix – Note 2**)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:40)	**Notation**	– Mention that the general formula $$\frac{n(n-1)}{2}$$ is called "N choose 2" – Demonstrate to students that the total number of handshakes for 10 people can be written as $\binom{10}{2}$ or 10C2 (it may be useful to act this out before-hand) – Explain how to compute this on the calculator (press "10", "shift" "nCr" and then "2") – You may wish to ask students what the total number of handshakes would be for a larger group of people (e.g. 120 people)
10 mins (00:50)	**Activity 3** **Pizza Toppings**	– **Activity Sheet 3**: Students are asked to find the total number of unique pairs of pizza toppings (see **Appendix – Note 3**).
5 mins (00:55)	**Combinatorics**	– Mention that the concepts covered in the workshop relate to combinatorics and explain what is meant by this term – Outline some of the applications of combinatorics (see **Workshop Introduction**)

The Handshake Puzzle: Workshop Appendix

Note 1: Solutions for Activity 1

Q1. (i) What is the maximum number of handshakes between the following numbers of people?

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES	DIAGRAM
Two	1	
Three	3	
Four	6	
Five	10	
Six	15	Same idea as above but with 15 lines i.e. each person shakes hands with 5 other people

(ii) Identifying the pattern as quadratic (link to curriculum - Strand 3: Numbers)

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES	DIFFERENCE BETWEEN ROWS: 1ST DIFFERENCE	DIFFERENCE BETWEEN ROWS: 2ND DIFFERENCE
Two	1	+2	+1
Three	3	+3	+1
Four	6	+4	+1
Five	10	+5	
Six	15		

(iii) How many possible handshakes are there between each of the following?

- Nine people = 36 handshakes
- Ten people = 45 handshakes

Note 2: Solutions for Activity 2

Q2. (i) How does Bob's answer compare to your answer in Q1 (iii)?

He multiplied our answer by 2 and therefore double counted the number of handshakes i.e. each pair of people shook hands twice according to Bob's calculations.

(ii) Using Bob's calculations, what could we do to his answer to get the correct number of handshakes in the room?

$$\frac{10(9)}{2} = 45$$

(iii) Would the above method work if Bob was in the room with 8 other people (9 in total)?

Yes:

$$\frac{9(9-1)}{2} = \frac{9(8)}{2} = 36$$

(iv) What if there were 5 other people in the room (6 in total)?

Using the above method, we get

$$\frac{6(6-1)}{2} = 15$$

Compare this with the answer for 6 people in question 1 (i)

(v) Given that n is the number of people in the room, can you find a general formula for the total number of handshakes?

$$\frac{n(n-1)}{2}$$

Explanation: If there are n people in a room, each person will shake hands with the (n - 1) remaining people as they will not be shaking hands with themselves – hence we get (n)(n - 1). However, we need to divide this number by 2 since one handshake allows two people to shake hands.

Note 3: Solutions for Pizza Problem

Q1. (i) How many unique toppings can we have for Topping 1?

4 unique toppings

(ii) Write out all the pairs of toppings that you could have on your pizza:

{Ham, Tomato}	{Ham, Pineapple}	{Ham, Olives}
{Pineapple, Tomato}	{Pineapple, Olives}	~~{Pineapple, Ham}~~
{Tomato, Olives}	~~{Tomato, Ham}~~	~~{Tomato, Pineapple}~~
~~{Olives, Ham}~~	~~{Olives, Pineapple}~~	~~{Olives, Tomato}~~

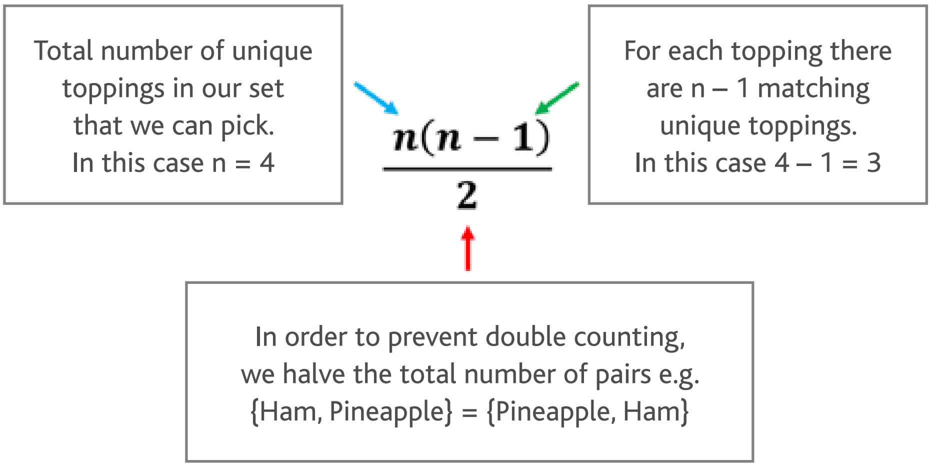
(iii) How many unique pizzas with only two toppings are there in total?

The order of the toppings does not matter on a pizza, hence there are 6 unique pizzas (crossing out the doubles as above) i.e. {Ham, Pineapple} = {Pineapple, Ham} etc.

Q2. (i) What is the value of n in this activity?

Since there are 4 toppings, the value of n is 4.

(ii) What does each part of the formula $\frac{n(n-1)}{2}$ represent based on the activity we just did?



(iii) Using the formula, calculate the number of unique pairs of toppings you can construct from the set {Ham, Pineapple, Tomato, Olives} and check your answer with a calculator.

$$\frac{4(4-1)}{2} = 6$$

To verify your answer on the calculator (instructions for Casio), press "4", then "shift", then "nCr" and then "2".

Sources and Additional Resources:

http://illuminations.nctm.org/Activity.aspx?id=6756 (Handshake puzzle simulator)

http://world.mathigon.org/Combinatorics (Combinatorics)

The Handshake Puzzle: Activity 1

Q1. (i) What is the maximum number of handshakes between the following numbers of people? (Remember the rules of the handshake puzzle.)

(Hint: you may like to draw a diagram and/or simulate the problem in small groups.)

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES	ROUGH WORK/DIAGRAM
Two		
Three		
Four		
Five		
Six		

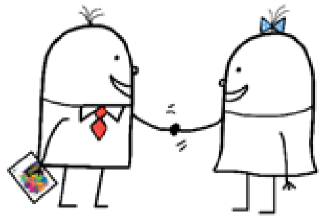
(ii) Can you identify a pattern between the number of people and the number of handshakes?

(Hint: it may be helpful to draw a table.)

(iii) How many possible handshakes are there between each of the following?

Nine people

Ten people



The Handshake Puzzle: Activity 2

Q2. Bob was in a room with 9 other people and he shook hands with each of them exactly once, as did every other person in the room. To find the total number of handshakes, Bob multiplied 10 by 9 and got 90.

(i) How does Bob's answer compare to your answer in Q1 (iii)?

(ii) Using Bob's calculations, what could we do to his answer to get the correct number of handshakes in the room?

(iii) Would the above method work if Bob was in the room with 8 other people (9 in total)?

(iv) What if there were just 5 other people in the room (6 in total)?

(v) Given that n is the number of people in the room, can you find a general formula for the total number of handshakes?

The Handshake Puzzle: Activity 3

You are ordering a pizza and can pick two toppings from the following set:

{Ham, Pineapple, Tomato, Olives}

Q1. We want to find out how many unique pizzas with only two toppings can be made from this set.

Rules:

- The pizza must have two toppings
- The pizzas are chosen by {Topping 1, Topping 2}
- We ***cannot*** use the same topping twice e.g. {Ham, Ham}

(i) How many unique toppings can we have for Topping 1?

(ii) Write out all the pairs of toppings that you could have on your pizza:

(iii) How many **unique** pizzas with only two toppings are there in total?

Q2. Considering our formula from before:

(i) What is the value of n in this activity?

(ii) What does each part of the formula $\frac{n(n-1)}{2}$ represent based on the activity we just did?

Write your answers in the boxes below.

$$\frac{n(n-1)}{2}$$

(iii) Using the formula, calculate the number of unique pairs of toppings you can construct from the set {Ham, Pineapple, Tomato, Olives} and check your answer with a calculator.